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The effect of non-uniformity of modulated wavepackets on the mechanism of Benjamin–Feir instability

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The effect of non-uniformity on the development of a modulated, weakly nonlinear wavepacket is studied. The non-uniformity, characterized by slowly varying wavenumber and frequency of the primary wave, may lead to significant modification of the stability properties compared with the uniform case. As a specific example we consider a modulated Stokes wave on deep water. In the uniform case such a wave proves to be definitely unstable (Benjamin & Feir 1967). In the non-uniform case, on the other hand, the wave may become stable under certain conditions. One of these is an increase of the local group velocity in the direction of wave propagation. Then the Benjamin–Feir instability mechanism is quenched on a time scale determined by the degree of non-uniformity. In addition, a sufficient degree of non-uniformity leads to stability of the wave to linear perturbations. However, when the local group velocity decreases in the direction of wave propagation, non-uniformity has a destabilizing effect. A comparison is made with experiments. It is also shown that the analysis, based on this specific example, is readily applied to a greater variety of non-uniform, dispersive waves.

1. Introduction

Benjamin (1967) and Benjamin & Feir (1967) demonstrated the instability of a uniform Stokes wave train due to modulations. Rather than follow their original analysis, Hasimoto & Ono (1972), Yuen & Lake (1975) and Stuart & DiPrima (1978) gave an alternative treatment, leading to the same results, but based on a nonlinear evolution equation for the envelope of the Stokes wave. Starting from this so-called nonlinear Schrödinger equation, the latter authors were able to establish the full connection with the Benjamin–Feir instability mechanism and described the main features of this in terms of a synchronous resonance. The extension to three-dimensional packets of surface gravity waves was studied by Davey & Stewartson (1974). Benney & Newell (1967) were the first to use the Schrödinger equation as a basis for the analysis of weakly non-linear, uniform waves.

Based on approximation methods used in geometrical optics and WKB-theory, Whitham (1965*a, b*) developed a general theory of non-uniform waves. In particular, he generalized the concept of wavenumber and frequency by defining a phase function whose time- and space-derivatives are allowed to depend on slow variables. Thus, in addition to the wave amplitude, the wavenumber and frequency also vary slowly in space and time. For a detailed discussion the reader is referred to Whitham's monograph (1974).

It is the purpose of the present paper to study the effect of non-uniformity on the development of a modulated wave. Based on Whitham's theory, and using the method of multiple scales, we derive a nonlinear Schrödinger equation, modified by the effect of non-uniformity. As a preliminary it will be useful to consider the equation.

$$\frac{\partial^2\psi}{\partial t^2} - d^2\frac{\partial^2\psi}{\partial x^2} + b^2\psi + a\psi^3 = 0, \tag{1.1}$$

where a, b and d are real constants. This turns out to be a representative model of weakly nonlinear, dispersive waves. When a small parameter ϵ is a measure of the amplitude of the wave, relevant slow variables are

$$x_n = \epsilon^n x, \quad t_n = \epsilon^n t, \quad n = 1, 2, \ldots \,. \tag{1.2}$$

The phase function θ is defined by the pair of equations

$$\frac{\partial\theta}{\partial x} = k(x_2, t_2), \quad \frac{\partial\theta}{\partial t} = -\omega(x_2, t_2), \tag{1.3}$$

where k and ω are the local wavenumber and frequency of the primary wave. These are related according to the consistency relation

$$\frac{\partial k}{\partial t_2} + \frac{\partial\omega}{\partial x_2} = 0. \tag{1.4}$$

The degree of non-uniformity, or modulation depth, is determined by the particular choice of the slow variables in (1.3).

As a solution of equation (1.1) we take

$$\psi = \epsilon\{A(x_1, x_2, t_1, t_2)\,\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.} + O(\epsilon^3), \tag{1.5}$$

where c.c. denotes the complex conjugate. This leads to the equation

$$2\mathrm{i}\omega\left\{\frac{\partial A}{\partial t_2} + c_g\frac{\partial A}{\partial x_2} + \frac{1}{2}\frac{\partial c_g}{\partial x_2}A\right\} + \frac{b^2d^2}{\omega^2}\frac{\partial^2 A}{\partial x_1^2} = 3a|A|^2A, \tag{1.6}$$

where c_g is the group velocity, determined by the dispersion relation $\omega^2 = k^2\,d^2 + b^2$.

Equation (1.6) includes the effects of dispersion, nonlinearity and non-uniformity (the term proportional to A). The last term follows from the particular choice of variables x_2 and t_2 in (1.3). When k and ω depend on slower variables, equation (1.6) reduces to the standard form of the nonlinear Schrödinger equation, although the co-efficients still depend on these variables. The case when k and ω depend on faster variables than x_2 and t_2 will not be considered. It is also noted that, in (1.3), the total variations of k and ω are $O(1)$, which implies a relatively broad wavenumber spectrum, with $\delta k/k = O(1)$.

For a fairly wide class of weakly nonlinear waves, whose energy is concentrated in a narrow band of wavenumbers, the slow variation in the complex amplitude can be described by the standard nonlinear Schrödinger equation (Benney & Newell 1967; Davey 1972; Kenneth 1974). In the non-uniform case, a simple extension applies. The only modification is the appearance of one extra term. When the amplitude equation depends on the slowest variables x_2 and t_2 only, the energy equation derived from (1.6) is of the form (2.11), with A_0 replaced by A. Thus, the energy balance prescribes the particular form of the aforementioned term. To a certain extent, this also explains why the terms between curly brackets in (1.6) and (2.9) are similar.

In §2 the modified nonlinear Schrödinger equation for a Stokes wave on deep water is derived. In §3 the development of the wave is described, supplied with the proper initial conditions on the frequency distribution and the wave envelope. The stability analysis is based on an equation for the linear perturbations. This describes the effect of non-uniformity on the mechanism of Benjamin–Feir instability. In § 4 we discuss the energetic aspects of this instability, and derive the equation governing the inter-action between the perturbations and the wave. Finally, in § 5 the results are discussed.

2. The modified Schrödinger equation

We choose a fixed Cartesian system of coordinates Oxz. The z-axis points vertically upwards, with $z = 0$ corresponding to the undisturbed free water surface. The x-axis is aligned with the propagation direction of a Stokes wavepacket. The water is irrotational, incompressible and deep with respect to the characteristic wavelength.

The potential ϕ is written as

$$\phi = \epsilon\{\alpha_1\,\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}\} + \epsilon^2\{\beta_0 + \beta_1\,\mathrm{e}^{\mathrm{i}\theta} + \beta_2\,\mathrm{e}^{2\mathrm{i}\theta} + \text{c.c.}\} + O(\epsilon^3). \tag{2.1}$$

Here α_1 and β_0, β_1, β_2 depend on the fast variable z and the slow variables (1.2), where ϵ is a small parameter, which is a measure of the amplitude of the wave. The phase function θ is defined by (1.3), which implies the consistency relation (1.4).

The boundary condition at infinite depth reads

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 \to 0, \quad z \to -\infty. \tag{2.2}$$

At the air–water interface, $z = \zeta$, we have the conditions

$$\frac{\partial\phi}{\partial z} = \frac{\partial\zeta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}, \tag{2.3}$$

$$2g\zeta + 2\frac{\partial\phi}{\partial t} + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 = 0, \tag{2.4}$$

where g is the gravitational acceleration.

The evolution equation for the envelope of the wave will be derived by the method of multiple scales. Referring to the work of Hasimoto & Ono (1972) for details of the procedure, only the principal results will be presented.

In a first-order theory, we obtain the dispersion relation for the primary wave, which is of the form $\omega^2 = gk$. With (1.4) this leads to the equation

$$\frac{\partial\omega}{\partial t_2} + c_g\frac{\partial\omega}{\partial x_2} = 0, \tag{2.5}$$

where $c_g = g/2\omega$ is the group velocity. Since the potential satisfies a Laplace equation, (2.2) implies that the coefficient α_1 in (2.1) may be written as

$$\alpha_1 = A(x_1, x_2, t_1, t_2)\,\mathrm{e}^{kz}, \tag{2.6}$$

where, to the order considered, the slower variables in (1.2), with $n > 2$, need not be taken into account. Then the expression for the surface elevation reads

$$g\zeta = \epsilon\{\mathrm{i}\omega A\,\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}\} + O(\epsilon^2), \tag{2.7}$$

derived from (2.3), (2.4) and (2.6).

A second-order theory leads to the equation

$$\frac{\partial A}{\partial t_1} + c_g \frac{\partial A}{\partial x_1} = 0. \tag{2.8}$$

This implies that the envelope A of the wave propagates with the group velocity, where the shape of this remains unchanged initially.

In a third-order theory, which also describes the dependence of the envelope on the slower variables x_2 and t_2, we obtain

$$2\mathrm{i}\omega \left\{ \frac{\partial A}{\partial t_2} + c_g \frac{\partial A}{\partial x_2} + \frac{1}{2}\frac{\partial c_g}{\partial x_2} A \right\} - \frac{g}{4k}\frac{\partial^2 A}{\partial x_1^2} = 4k^4 |A|^2 A. \tag{2.9}$$

This so-called modified Schrödinger equation includes the effect of non-uniformity as described in § 1. A similar equation was previously derived by Djordjevic & Redekopp (1978) and Turpin, Benmoussa & Mei (1983). These authors studied the effect of slowly varying depth on the development of a modulated Stokes wavepacket. Thus, a varying depth induces non-uniformity of the Stokes wave in this case.

In the uniform case, when k and ω are constants, the term proportional to A vanishes, and (2.9) reduces to the standard form of the Schrödinger equation for a deep-water Stokes wave.

In what follows we study the stability of the so-called fundamental wave, denoted by A_0. This depends on the slow variables x_2 and t_2 only, and is described by the equation

$$2\mathrm{i}\omega \left\{ \frac{\partial A_0}{\partial t_2} + c_g \frac{\partial A_0}{\partial x_2} + \frac{1}{2}\frac{\partial c_g}{\partial x_2} A_0 \right\} = 4k^4 |A_0|^2 A_0. \tag{2.10}$$

Then the equation

$$\frac{\partial}{\partial t_2}\{|A_0|^2\} + \frac{\partial}{x_2}\{c_g |A_0|^2\} = 0, \tag{2.11}$$

derived from (2.10), shows that the energy of the fundamental wave propagates with the local group velocity.

3. Stability analysis

Depending on whether the group velocity increases or decreases in the direction of propagation, the associated wave will be called expansive or compressive (Whitham 1974). In the former case, the solution of equation (2.5) remains one-valued because of the diverging characteristics. In the latter case, this solution will become multi-valued after a finite time due to the converging characteristics.

3.1. *The expansive wave*

We start from equation (2.5), with the initial condition

$$\omega = \sigma(x_2) \quad (t_2 = 0). \tag{3.1}$$

Then the method of characteristics leads to a solution of the form

$$\omega = \sigma(\eta), \quad \eta = x_2 - (g/2\sigma(\eta))t_2. \tag{3.2}$$

The dispersion relation implies that $c_g = g/2\sigma(x_2)$ initially. Introducing the notation $c(x_2) = g/2\sigma(x_2)$, it is found that $c_g = c(\eta)$ for $\tau \geqslant 0$, in view of (3.2). The variable η represents a reference frame moving with the group velocity. With $c' = \mathrm{d}c/\mathrm{d}\eta > 0$ for an expansive wave, this variable is uniquely determined by x_2 and t_2, which implies a one-valued, unique solution (3.2).

It will be convenient to introduce the transformations

$$\eta = x_2 - c(\eta)t_2, \quad \xi = x_1 - c(\eta)t_1, \quad \tau = t_2. \tag{3.3}$$

Combined with equation (2.8) this implies that A depends only on the new variables defined by (3.3). Then equation (2.9) transforms into

$$2\mathrm{i}\omega\frac{\partial A}{\partial \tau} - \frac{g}{4k}\frac{\partial^2 A}{\partial \xi^2} + \mathrm{i}\omega\left(\frac{c'}{1+c'\tau}\right)A = 4k^4|A|^2A, \tag{3.4}$$

with $\omega = \sigma(\eta)$, $gk = \sigma^2(\eta)$.

The solution of equation (3.4) is written in the form

$$A = A_0(\eta,\tau)\{1 + \mu B(\xi,\eta,\tau;\mu)\}, \tag{3.5}$$

where μ is a small parameter. This implies that

$$2\mathrm{i}\omega\frac{\partial A_0}{\partial \tau} + \mathrm{i}\omega\left(\frac{c'}{1+c'\tau}\right)A_0 = 4k^4|A_0|^2A_0, \tag{3.6}$$

which is the transformed version of (2.10). Thus, the wave described by (3.5) represents a small-amplitude perturbation of the fundamental wave $A_0(\eta, \tau)$.

The perturbation B in (3.5) satisfies the equation

$$2\mathrm{i}\omega\frac{\partial B}{\partial \tau} - \frac{g}{4k}\frac{\partial^2 B}{\partial \xi^2} = 4k^4|A_0|^2\{B + B^* + \mu(B^2 + 2|B|^2) + \mu^2|B|^2B\}, \tag{3.7}$$

where the asterisk denotes the complex conjugate.

The fundamental wave is written as $A_0 = |A_0|\exp(\mathrm{i}\varphi)$, where φ is the phase. From equation (3.6) we then obtain

$$\frac{\partial}{\partial \tau}|A_0| = -\left\{\frac{c'|A_0|}{2(1+c'\tau)}\right\}. \tag{3.8}$$

The initial condition

$$A_0 = K_0(x_2) \quad (\tau = 0) \tag{3.9}$$

leads to the unique solution

$$|A_0|^2 = \frac{|K_0(\eta)|^2}{1+c'\tau}, \tag{3.10}$$

where it is required that K_0 is bounded as $x_2 \to \pm\infty$. When K_0 vanishes in these limits, A_0 represents the envelope of a wavepacket. The nonlinear term in equation (3.6) only affects the phase.

Next we return to (3.7) and introduce the perturbation expansion

$$B = B_0(\xi,\eta,\tau) + \mu B_1(\xi,\eta,\tau) + O(\mu^2). \tag{3.11}$$

Then the equation for B_0 reads

$$2\mathrm{i}\omega\frac{\partial B_0}{\partial \tau} - \frac{g}{4k}\frac{\partial^2 B_0}{\partial \xi^2} = 4k^4|A_0|^2(B_0 + B_0^*). \tag{3.12}$$

Combined with (3.10) this determines the linear stability characteristics of the wave.

The wave will be called stable to linear perturbations if $B_0 = O(1)$ as $\tau \to \infty$. In view of (3.5) and (3.11) this definition implies that, on long time scales, there is no substantial growth of the perturbation with respect to the fundamental wave.

Following Stuart & DiPrima (1978) we seek normal modes of the form

$$B_0 = F(\eta,\tau)\,\mathrm{e}^{\mathrm{i}r\xi} + \text{c.c.} + \mathrm{i}\{G(\eta,\tau)\,\mathrm{e}^{\mathrm{i}r\xi} + \text{c.c.}\}, \tag{3.13}$$

where r is a real wavenumber. Introducing the dimensionless variables

$$\tilde{\tau} = \omega\tau, \quad \gamma = \frac{r^2}{8k^2}, \quad s = \frac{c'(\eta)}{\omega}, \tag{3.14}$$

making use of (3.10), (3.12), (3.13), and omitting the tilde, we obtain the pair of equations

$$\frac{\partial F}{\partial \tau} = -\gamma G, \tag{3.15}$$

$$\frac{\partial G}{\partial \tau} = \gamma(1 - a_0^2)F, \quad a_0^2 = \frac{(ak/\epsilon)^2}{\gamma(1 + s\tau)}, \tag{3.16}$$

where ak is the initial wave steepness, with $ak = O(\epsilon)$. Then F satisfies the single equation

$$\frac{\partial^2 F}{\partial \tau^2} + qF = 0, \quad q = \gamma^2(1 - a_0^2). \tag{3.17}$$

The reference amplitude a_0 is a measure of the amplitude of the fundamental wave, with $a_0^2 = (4k^4/\omega^2\gamma)|A_0|^2$, and $|A_0|^2$ given by (3.10).

Equation (3.17) may also be written in the dimensional form

$$\frac{\partial^2 F}{\partial \tau^2} + \omega^2\gamma^2 F = 4\gamma k^4 |A_0|^2 F,$$

which allows a more direct comparison with previous work.

In the uniform case the parameter s in (3.14) vanishes, which implies a time-independent reference amplitude and a time-independent coefficient q in (3.17). Then F grows exponentially with time if

$$\left(\frac{ak}{\epsilon}\right)^2 > \frac{r^2}{8k^2}. \tag{3.18}$$

This agrees with the condition for Benjamin–Feir instability. It also implies that $r/k = O(1)$.

We now first give a brief and qualitative description of the modified wave behaviour in comparison with the uniform case. The condition (3.18) for initial growth of the perturbation remains the same. Due to the steadily decreasing amplitude of the fundamental wave, however, the coefficient q of equation (3.17), which should be negative initially in order that growth occurs, will change sign at $\tau = \tau_0$, say. This leads to a transition from growth to oscillatory behaviour, and implies that the Benjamin–Feir instability is quenched. Crudely speaking, this occurs at the cutoff $\tau = \tau_0$. In other words, quenching occurs as soon as the amplitude of the fundamental wave drops below a critical threshold value, corresponding to $a_0 < 1$. Whether the perturbation grows or oscillates also depends on the location within the wavepacket.

From (3.15) and (3.16) we obtain

$$\frac{\partial}{\partial \tau}\{|F|^2 + |G|^2\} = a_0^2 \frac{\partial}{\partial \tau}\{|F|^2\}. \tag{3.19}$$

Furthermore, in view of (3.13),

$$\overline{B_0 B_0^*} = 2(|F|^2 + |G|^2), \tag{3.20}$$

where the overbar denotes averaging over a wavelength of the perturbation. Equation (3.17) implies that

$$\frac{\partial^2}{\partial \tau^2}\{|F|^2\} = -2q|F|^2 + 2\left|\frac{\partial F}{\partial \tau}\right|^2. \tag{3.21}$$

Now suppose that $\partial\{\overline{B_0 B_0^*}\}/\partial\tau = 0$ at $\tau = \tau_+$, say, with $0 < \tau_+ < \tau_0$. From (3.19)–(3.21), with $q < 0$, it then follows that $\partial^2\{\overline{B_0 B_0^*}\}/\partial\tau^2 > 0$ at $\tau = \tau_+$, which implies a local minimum of $\overline{B_0 B_0^*}$ at this point in time. From this we conclude that, if $\partial\{\overline{B_0 B_0^*}\}/\partial\tau > 0$ initially, this derivative remains positive as long as the coefficient q is negative. In other words, in front of the cutoff ($\tau < \tau_0$) the mechanism of Benjamin–Feir instability is active, and the amplitude of the perturbation grows. Behind the cutoff ($\tau > \tau_0$) some further growth will occur as long as the right-hand side of (3.21) is positive.

Introducing the new variable $z = (2\mathrm{i}\gamma/s)(1 + s\tau)$, equation (3.17) transforms into Whittaker's equation. Then both F and G may be expressed in terms of Kummer functions (Abramowitz & Stegun 1964). From the known asymptotic behaviour of these functions we then find that the perturbation B_0, defined by (3.13), is bounded as $\tau \to \infty$. This implies that, formally at least, the wave is stable. From (3.15), (3.17) and (3.20) it also follows that

$$\overline{B_0 B_0^*} \to \text{const.} \quad \text{as} \quad \tau \to \infty, \tag{3.22}$$

which implies a constant r.m.s. amplitude of the perturbation in this limit.

The parameter s is (3.14) is a measure of local non-uniformity. To study the effect of weak non-uniformity, we take $s = \delta$, with $\epsilon \ll \delta \ll 1$ and fixed η. Then equation (3.17) may be transformed into an equation of the form $\mathrm{d}^2F/\mathrm{d}x^2 + \lambda^2 r(x)F = 0$, with $x = \tau/\lambda$ and $\lambda = 1/\delta \gg 1$. Using Langer's transformation method (Nayfeh 1973), one obtains a leading-order solution, which is uniformly valid for all $x \geqslant 0$, including the cutoff $x = x_0$, with $x_0 = \delta\tau_0 = O(1)$. Behind the cutoff we have

$$F = c_1\{r(x)\}^{-1/4} \sin\left\{\lambda \int_{x_0}^{x} \{r(u)\}^{1/2}\,\mathrm{d}u + \tfrac{1}{4}\pi\right\} \quad (x > x_0) \tag{3.23}$$

with

$$c_1 = C_1 \exp\left\{\lambda \int_{0}^{x_0} \{-r(u)\}^{1/2}\,\mathrm{d}u\right\}, \tag{3.24}$$

where $C_1 = O(1)$ in order that F and $\mathrm{d}F/\mathrm{d}\tau$ are $O(1)$ initially.

This shows that, in the case of weak non-uniformity, the amplitude of the perturbation becomes large near the cutoff and remains so for $\tau > \tau_0$. The reason for this is the relatively long time scale on which the Benjamin–Feir instability is active, which permits growth of the perturbation for a long period of time.

The case of strong non-uniformity corresponds to $s = 1/\delta$, with $\epsilon \ll \delta \ll 1$. Introducing the transformation $z = \tau/\delta$, and keeping η fixed, equation (3.17) becomes of the form

$$\frac{\mathrm{d}^2F}{\mathrm{d}z^2} + \delta^2 r(z)F = 0.$$

This is the equation for the so-called inner layer, defined by $z = O(1)$. The leading-order equation for the outer layer, defined by $\tau = O(1)$, reads

$$\frac{\mathrm{d}^2F}{\mathrm{d}\tau^2} + \gamma^2 F = 0,$$

which results from equation (3.17) in the limit $\delta \to 0$, τ fixed. The initial conditions, given by $F = \alpha$, $\mathrm{d}F/\mathrm{d}\tau = \beta$ at $\tau = 0$, determine the inner-layer solution according to $F = \alpha + \delta\beta z$. Matching this asymptotically with the outer-layer solution, we obtain the leading-order expression

$$F = \alpha \cos \gamma\tau + \frac{\beta}{\gamma} \sin \gamma\tau. \tag{3.25}$$

Thus, in the case of strong non-uniformity, (3.25) shows that both the amplitude and the frequency of the perturbation are independent of time. Apparently, the Benjamin–Feir instability is completely suppressed in this case.

From the above examples we conclude that a sufficient degree of non-uniformity leads to stability of the wave to linear perturbations. In that case growth is quenched after a relatively short period of time, which limits the amplitude of the perturbation. Figure 1 shows the r.m.s. amplitude of the perturbation against the dimensionless time for various values of the parameter s, defined in (3.14), and fixed initial conditions. Furthermore, $\gamma = 2$ and $ak/\epsilon = 1.4$, where ak is the initial wave steepness. The maximum amplitude seems to decrease with increasing degree of non-uniformity, which also indicates the stabilizing effect of non-uniformity.

It is recalled that the validity of these results is restricted to the case of an expansive wave. Such a wave could be generated, for instance, by a wave maker with steadily increasing frequency. Then the group velocity of the Stokes wave increases with the distance from the source.

3.2. *The compressive wave*

The amplitude of the fundamental wave changes in space and time due to a combined effect of dispersion and non-uniformity. In the case of a compressive wave, the amplitude may become large. To see this, we start from the energy equation (2.11) for the fundamental wave. From this equation it follows that

$$\int_a^b |A_0(x_2, t_2)|^2 \,\mathrm{d}x_2 = \text{const.}, \tag{3.26}$$

where $a = a(t_2)$ and $b = b(t_2)$ move with the local group velocity (Whitham 1974). If the domain of integration is small, this implies that the local amplitude steadily increases with time in regions where the wave is compressive. This is also expressed by (3.10) which, however, predicts that the amplitude tends to infinity in a finite time. Although this restricts the validity of (3.10), the amplitude may become very large indeed. To show this, we start from the model equation

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} + \psi = 0, \tag{3.27}$$

which describes a linear, dispersive wave. The equation is taken linear because it may be expected that the effect of nonlinearity is of higher order. Indeed, when a term proportional to ψ^3 is included, for instance, the equation for the amplitude of the fundamental wave proves to be of the form (2.10). As expressed by this equation, the

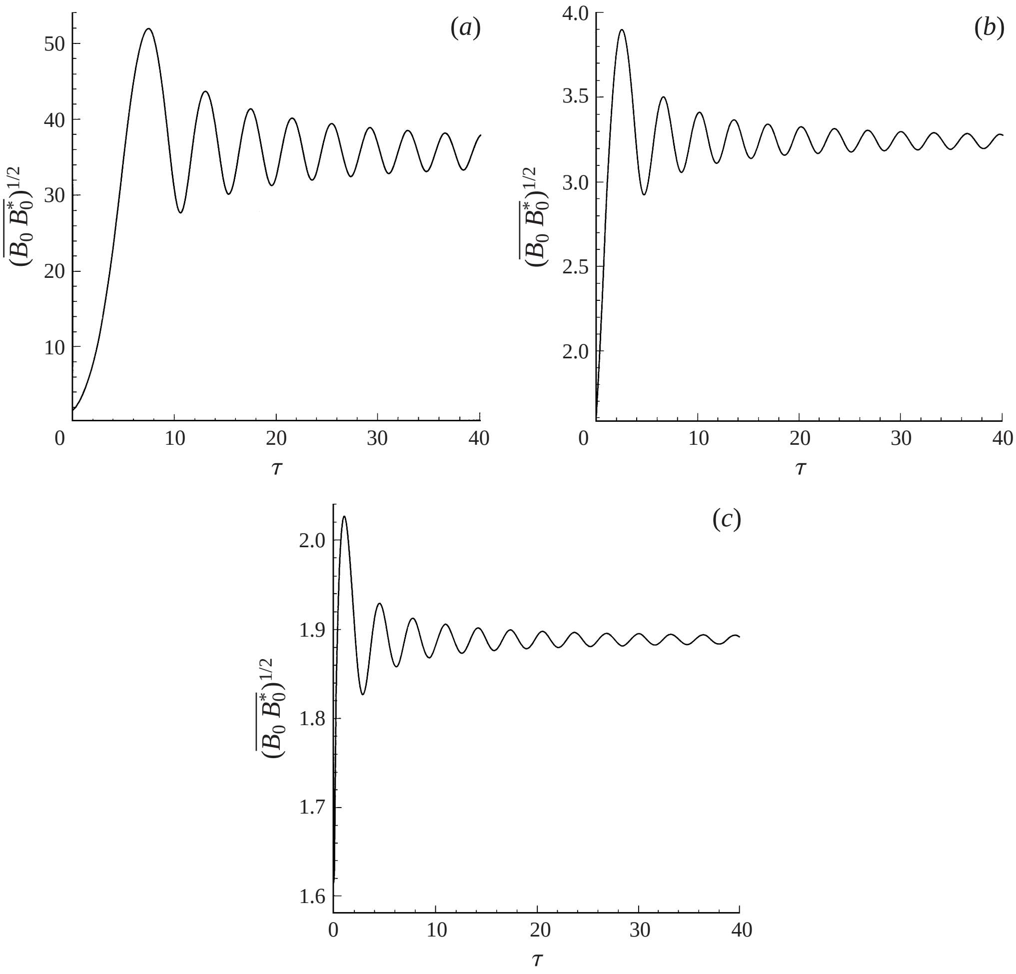

FIGURE 1. The r.m.s. amplitude of the perturbation against the dimensionless time for (a) $s = \frac{1}{5}$, (b) $s = 1$, (c) $s = 5$. Initial conditions: $F = 1$, $\partial F/\partial \tau = \frac{1}{2}$ at $\tau = 0$.

nonlinear term only affects the phase of the wave. In the Appendix it is shown that the amplitude becomes very large indeed when the sufficient condition (A 15) is satisfied.

Analogy with the present problem implies that the amplitude of the fundamental wave, approximated by (3.10), will become large under the same restriction. The rate of growth of the wave amplitude increases with increasing degree of local non-uniformity, as may be inferred from the equation $1 + c'(\eta)\tau = 0$. Thus, in the case of relatively strong non-uniformity, the amplitude of the wave will soon become large. In the case of weak non-uniformity, on the other hand, the steadily growing amplitude of the wave will become large on a much longer time scale. In addition, the Benjamin–Feir instability is not quenched in this case due to the compressive nature of the wave. This implies that the amplitude of the perturbation may become large as well due to the cumulative effect of the instability. We conclude that, in the case of a compressive wave, non-uniformity has a destabilizing effect.

4. Higher-order theory and energetic aspects

We start from the equation

$$2\mathrm{i}\omega\frac{\partial B_1}{\partial\tau}-\frac{g}{4k}\frac{\partial^2 B_1}{\partial\xi^2}-4k^4|A_0|^2(B_1+B_1^*)=4k^4|A_0|^2(B_0^2+2|B_0|^2), \tag{4.1}$$

obtained from (3.7) and (3.11). The solution determines the higher-order correction proportional to μ^2 in (3.5).

Instead of (3.13) it will be convenient to write B_0 in the form

$$B_0=C(\eta,\tau)\,\mathrm{e}^{\mathrm{i}r\xi}+D(\eta,\tau)\,\mathrm{e}^{-\mathrm{i}r\xi}. \tag{4.2}$$

Then B_1 has a non-zero, horizontally averaged part $\overline{B}_1$, given by

$$\overline{B}_1=Q_0(\eta,\tau)+\mathrm{i}Q_1(\eta,\tau), \tag{4.3}$$

where Q_0 and Q_1 are real. From (4.1) and (4.2) it then follows that

$$\omega\frac{\partial Q_0}{\partial\tau}=4k^4|A_0|^2\,\mathrm{Im}\,CD, \tag{4.4}$$

where Im denotes the imaginary part.

Substitution of (4.2) into (3.12) leads to the pair of equations

$$2\mathrm{i}\omega\frac{\partial C}{\partial\tau}=-h^2C+R_0(C+D^*), \tag{4.5}$$

$$2\mathrm{i}\omega\frac{\partial D^*}{\partial\tau}=h^2D^*-R_0(C+D^*), \tag{4.6}$$

where $R_0=4k^4|A_0|^2$ and $h^2=gr^2/4k$. This implies that

$$\frac{\partial}{\partial\tau}\{|C|^2\}=\frac{\partial}{\partial\tau}\{|D|^2\}=-\frac{R_0}{\omega}\,\mathrm{Im}\,CD. \tag{4.7}$$

From (4.2), (4.4) and (4.7) we obtain

$$\frac{\partial Q_0}{\partial\tau}=-\frac{1}{2}\frac{\partial}{\partial\tau}\{\overline{B_0B_0^*}\}. \tag{4.8}$$

From (3.5), (3.11), (4.2) and (4.3) it follows that

$$\begin{aligned}|\overline{A}|^2&=|A_0|^2\{1+2\mu^2Q_0+o(\mu^2)\}\\&=E_0+|A_0|^2o(\mu^2),\quad E_0=|A_0|^2(1+2\mu^2Q_0),\end{aligned} \tag{4.9}$$

where $\overline{A}$ is the horizontally averaged, complex amplitude of the wave, modified by non-linear interactions. At leading order, E_0 is proportional to the energy of the modified wave. Then (3.3), (3.8) and (4.8) imply the equation

$$\begin{aligned}\frac{\partial E_0}{\partial t_2}+\frac{\partial}{\partial x_2}\{c_gE_0\}&=\frac{\partial E_0}{\partial\tau}+\left(\frac{c'}{1+c'\tau}\right)E_0\\&=-\mu^2|A_0|^2\frac{\partial}{\partial\tau}\{\overline{B_0B_0^*}\},\end{aligned} \tag{4.10}$$

which describes the energy transfer between the wave and the perturbations. This equation implies that energy is extracted from the wave if the r.m.s. amplitude of the perturbation increases with time.

The perturbation is of the form (3.13), where F and G satisfy the equation (3.15) and (3.17). From (3.19) and (3.20) it follows that the time-derivatives of $|F|^2$ and $\overline{B_0B_0^*}$

have the same sign, which means that the direction of energy transfer is determined by the sign of $\partial |F|^2/\partial \tau$ as well. Thus, when this derivative is negative, energy is transferred from the perturbation to the wave. Figure 1 shows that this reversal may occur behind the cutoff of equation (3.17).

In the case of strong non-uniformity, treated below (3.24), F is of the form (3.25). From (3.19) and (3.20) it then follows that $\overline{B_0 B_0^*}$ is independent of time. In view of (4.10) this implies that there is no interaction between the wave and the perturbation in this limiting case. For the general case, we have shown that $\overline{B_0 B_0^*}$ tends to a constant as $\tau \to \infty$; cf. (3.22). This implies that, in the case of weaker non-uniformity, the interaction is small only after a sufficiently long period of time.

5. Discussion of results

We have studied the effect of non-uniformity on the mechanism of Benjamin–Feir instability. At a sufficient degree of non-uniformity, or sufficiently deep modulation, the principal result is that the stability properties of the wave are modified substantially compared with the uniform case. As a specific example we considered a deep-water Stokes wave, but this restriction seems not essential.

When the local group velocity increases in the direction of wave propagation, the Benjamin–Feir instability is quenched on a time scale determined by the degree of non-uniformity. This implies that the amplitude of the wave perturbations remains bounded as time increases. If the non-uniformity is strong enough, the perturbations remain so small that the wave is actually stable. Thus, in the case of deep-water Stokes wave, generated by a wave maker with steadily increasing frequency, it is expected that the wave is stable if the rate of change of the frequency is high enough. Non-uniformity of a Stokes wave may also be induced by slowly varying depth, see e.g. Turpin *et al.* (1983).

When the local group velocity decreases in the direction of wave propagation, non-uniformity has a destabilizing effect. In the case of strong non-uniformity there is a rapid growth of the fundamental wave, which may lead to wave breaking on a short time scale. In the case of weak non-uniformity, the amplitude of the perturbation may become large because the Benjamin–Feir instability is not quenched.

The degree of non-uniformity varies slowly along the wave. The compressive and expansive parts of the wave propagate with the local group velocity, and the rate of compression or expansion depends on the degree of local non-uniformity. Stability of the wave may only occur in the expansive parts.

The concept of uniformity of a modulated wavepacket is an idealization. It implies that the form of the propagating wave envelope does not change. In reality, however, a certain degree of non-uniformity is always present, characterized by amplitude dispersion. This is also observed in the experiments of Pierson, Donelan & Hui (1992). These experiments on water wave groups show deformation of the propagating wave envelopes. The width of the envelopes increases with the distance from the wave maker. This is especially clear from their figure 3(a), which also reveals the corresponding property of increasing group velocity in the direction of wave propagation. Since wave instabilities were not observed, this provides some evidence for the validity of the present theory.

A restriction of this theory is the linear stability analysis, which excludes the effects of nonlinear perturbations. Another restriction is that the results only correspond to a limited class of initial conditions, expressed in terms of slowly varying wavenumber (or frequency), wave amplitude and phase. Nevertheless, the results seem physically relevant. However, experiments are necessary to verify the theory.

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Appendix. The compressive wave

The solution of equation (3.27) is assumed to be of the form

$$\psi = F_1(\theta, x_2, t_2) + \epsilon F_2(\theta, x_2, t_2) + \ldots, \tag{A 1}$$

where the independent variables are defined by (1.2) and (1.3). Then

$$\left\{ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 1 \right\} \psi = \{\mathrm{L} - \epsilon^2 \mathrm{M} - \epsilon^4 \mathrm{N}\} \psi, \tag{A 2}$$

with

$$\left. \begin{aligned} \mathrm{L} &= \omega^2 \frac{\partial^2}{\partial \theta^2} - k^2 \frac{\partial^2}{\partial \theta^2} + 1, \\ \mathrm{M} &= \left(2\omega \frac{\partial}{\partial t_2} + \frac{\partial \omega}{\partial t_2} + 2k \frac{\partial}{\partial x_2} + \frac{\partial k}{\partial x_2} \right) \frac{\partial}{\partial \theta}, \quad \mathrm{N} = \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial t_2^2}. \end{aligned} \right\} \tag{A 3}$$

The substitution (A 1) leads to the hierarchy of equations

$$\mathrm{L}F_1 = 0, \quad \mathrm{L}F_2 = 0, \tag{A 4}$$

$$\mathrm{L}F_3 = \mathrm{M}F_1, \quad \mathrm{L}F_4 = \mathrm{M}F_2, \tag{A 5}$$

$$\mathrm{L}F_{n+4} = \mathrm{M}F_{n+2} + \mathrm{N}F_n, \quad n = 1, 2, 3 \ldots . \tag{A 6}$$

The equations (A 4) for F_1 and F_2 have solutions of the form

$$(F_1, F_2) = (A_1(x_2, t_2), A_2(x_2, t_2))\, \mathrm{e}^{\mathrm{i}\theta}, \tag{A 7}$$

which implies the dispersion relation $\omega^2 = 1 + k^2$. The coefficients A_1 and A_2 in these expressions are determined from the equations

$$\mathrm{M}F_1 = 0, \quad \mathrm{M}F_2 = 0. \tag{A 8}$$

These result from (A 5) by the requirement of vanishing secular terms, which also implies that $\mathrm{L}F_3 = 0$ and $\mathrm{L}F_4 = 0$. The same requirement leads to the equations

$$\mathrm{M}F_{n+2} = \mathrm{N}F_n = 0, \quad n = 1, 2, 3 \ldots, \tag{A 9}$$

obtained from (A 6). Combined with previous results this implies

$$\mathrm{L}F_n = 0, \quad n = 1, 2, 3 \ldots . \tag{A 10}$$

Introducing the transformations (3.3), and making use of (1.4) and (2.5), the operators M and N in (A 3) may be written

$$\mathrm{M} = 2\mathrm{i}\omega \left\{ \frac{\partial}{\partial \tau} + \frac{1}{2} \left(\frac{c'}{1 + c'\tau} \right) \right\}, \tag{A 11}$$

$$\mathrm{N} = \left\{ \left(\frac{1}{1 + c'\tau} \right) \frac{\partial}{\partial \eta} \right\}^2 - \left\{ \frac{\partial}{\partial \tau} - \left(\frac{c}{1 + c'\tau} \right) \frac{\partial}{\partial \eta} \right\}^2 . \tag{A 12}$$

From (A 7), (A 8) and (A 11) we obtain

$$(A_1, A_2) = (K_1(\eta), K_2(\eta))(1 + c'\tau)^{-1/2}, \tag{A 13}$$

where K_1 and K_2 are determined by the initial conditions.

The equations (A 9) will be solved recursively in terms of the new variables η and τ, where use is made of (A 10)–(A 13). These equations can be solved separately for odd and even index n, corresponding to the series Σ_0 and Σ_e, respectively.

We consider the case of a compressive wave, for which $c'(\eta) < 0$. In the series Σ_0, which will be considered first, the first term, A_1, has a singularity proportional to $\{1 + c'(\eta)\tau\}^{-1/2}$ as $1 + c'(\eta)\tau \to 0$. From (A 9), with $n = 1$, it then follows that the next term, A_3, has a singularity proportional to $\{1 + c'(\eta)\tau\}^{-7/2}$ in this limit. Then the requirement $\epsilon^2 F_3/F_1 \ll 1$ for an asymptotic expansion Σ_0 implies the restriction

$$\epsilon^{2/3} \ll 1 + c'(\eta)\tau. \tag{A 14}$$

Proceeding in this way, it is found from the recurrence relation (A 9), with odd n, that the series Σ_0 is asymptotic if (A 14) is satisfied. This proves to be also true of the series Σ_e. We conclude that, for fixed η and ϵ, with $\epsilon \ll 1$, the first few terms in (A 1) represent the approximate solution for $\tau \geqslant 0$ as long as (A 14) is satisfied. This implies that the amplitude of the fundamental wave will become very large indeed if

$$\epsilon^{2/3} \ll 1 + c'(\eta)\tau \ll 1. \tag{A 15}$$

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